Math 249 Lecture 9 Notes

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1 Symmetric Functions and Young Diagrams

1.1 Symmetric functions

We want to systematically write out the character table for S_n, and the classical way, going back to Frobenius, is to introduce algebras of symmetric functions.

Definition 1.1. The *symmetric functions* are $\Lambda(x_1, \dots, x_n) = \mathbb{Z}[x_1, \dots, x_n]^{S_n}$. That is, they are polynomials in n variables with integer coefficients that are invariant under permutations of the variables $x_1, \dots, x_n$. We also define $\Lambda_{\mathbb{R}}(x_1, \dots, x_n) = \mathbb{R}[x_1, \dots, x_n]^{S_n}$.

There is a natural basis for these. Let $\lambda = (\lambda_1, \cdots \lambda_\ell)$ be an integer partition with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. Take

$$m_\lambda = \sum_{\sigma \in S_n} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(\ell)}^{\lambda_\ell}.$$

These m_λ functions form our basis.

Example 1.1. Some examples of the symmetric functions in this basis are

$$m_{(k)} = x_1^k + x_2^k + \cdots + x_n^k$$

$$m_{\underbrace{(1, 1, \dots, 1)}_{k}} = \sum_{\substack{I \subseteq [n] \\ |I| = k}} \prod_{i \in I} x_i.$$

Why are the m_λ a basis? They are linearly independent because no two distinct m_λ share any terms (look at the exponents of the x_i). And for the symmetric functions of degree at most N, there are enough enough linearly independent m_λ with $|\lambda| = \lambda_1 + \cdots \lambda_n = N$; in fact, $\{m_\lambda : |\lambda| = N\}$ spans each $\Lambda[x_1, \dots, x_n]_{(N)}$. and so the m_λ span the whole space.

We can define the symmetric functions for infinitely many variables, too. Let

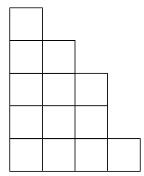
$$\Lambda_{\mathbb{R}} = \Lambda_{\mathbb{R}}(x_1, x_2, \dots),$$

where each symmetric function will have infinitely many terms.

1.2 Young diagrams

Definition 1.2. Given a partition λ, the *young diagram* of λ is the partition expressed as stacked rows of boxes.

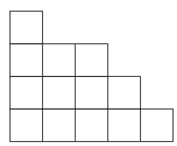
Example 1.2. Take $\lambda = (4, 3, 3, 2, 1)$. The Young diagram of λ is



Definition 1.3. Let λ be a partition. The *transpose* of λ, λ^*, is the partition $\lambda_i^* = |\{j : \lambda_j \geq i\}|$.

Much more intuitively, the Young diagram of λ^* is the transpose of the Young diagram of λ.

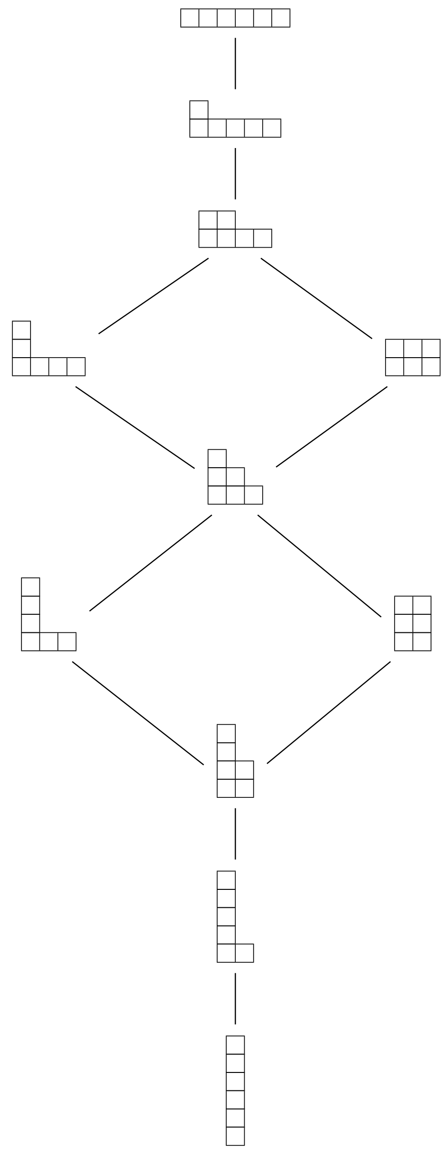
Example 1.3. Take $\lambda = (4, 3, 3, 2, 1)$, as before. Then the Young diagram of λ^* is:



So we can see that $\lambda^* = (5, 4, 3, 1)$.

Call $|\lambda| = \sum \lambda_i =$ number of boxes. We can form a partial ordering on $\{\lambda : |\lambda| = n\}$ by $\lambda \leq \mu$ iff $\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k$ for all k.

Example 1.4. Take the partitions of 6 $(4, 2)$, $(3, 2, 1)$, and $(3, 1, 1, 1)$. Then $(4, 2) \leq (3, 2, 1) \leq (3, 1, 1, 1)$. But we also have that $(4, 1, 1)$ is not comparable to $(3, 3)$. The partial order on partitions of 6 can be summarized in the following diagram:



1.3 Elementary symmetric functions

Definition 1.4. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition. The *elementary symmetric function* e_λ is defined by

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}, \text{where}\, e_k = m_{\underbrace{(1, 1, \ldots, 1)}_{k}}.$$

Proposition 1.1. *Let λ, μ be partitions of n. Then $e_\lambda = \sum_\mu a_{\lambda,\mu} m_\mu$, where*

1. *$a_{\lambda,\mu}$ = number of 0-1 matrices with row-sums $\lambda_1, \lambda_2, \ldots$ and column sums $\mu_1, \mu_2, \ldots$*
2. *$a_{\lambda,\lambda^*} = 1$*
3. *$a_{\lambda,\mu} = 0$ if $\mu \not\leq \lambda^*$.*

Example 1.5. For partitions of $n = 3$, we have

$$\begin{aligned} e_{(3)} &= m_{(1,1,1)}, \\ e_{(2,1)} &= m_{(2,1)} + 3m_{(1,1,1)}, \\ e_{(1,1,1)} &= m_{(3)} + 3m_{(2,1)} + 6m_{(1,1,1)}. \end{aligned}$$

Proof. (1) Since symmetric functions must be invariant under permuting the variables, to find the coefficient $a_{\lambda,\mu}$, we need only find the coefficient in front of a single monomial in each m_μ; we pick the monomial $x_1^{\mu_1} x_2^{\mu_2} \cdots x_\ell^{\mu_\ell}$. Each monomial in each e_{λ_i} will be a product of λ_i different variables, each with exponent 1; then we can find the coefficient of a monomial in e_λ by finding the number of ways to trace back where its x_i terms could have come from in the product of the e_{λ_i}.

Now consider the following matrix (represented as a table):

	μ_1	μ_2	μ_3	$\cdots$
λ_1	1	0	1	$\cdots$
λ_2	0	1	0	$\cdots$
λ_3	1	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We fill in this matrix as follows: Think of the column j as picking an x_j from some of $e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_\ell}$ (this is "where the x_j term came from" when you multiply the e_{λ_i}), and place a 1 in the space i, j if an x_j term comes from h_{λ_i} and a 0 otherwise. The product of all the x_j terms in a column should give $x_j^{\mu_j}$ (since this is the monomial in e_λ that we are looking at), so this is a matrix with column-sums μ_j. Similarly, each monomial in e_{λ_i} has λ_i variables, so the 1s in row i should add to be λ_i. We have produced a bijection between these matrices and the number of ways to get $x_1^{\mu_1} x_2^{\mu_2} \cdots x_\ell^{\mu_\ell}$ from the product of the e_{λ_i}, so the first part is proved.

(2) To prove the second assertion, construct the matrix above, where the 1s are placed in the pattern of the upside-down Young diagram of λ. For example, for $\lambda = (3, 2, 1, 1)$, construct the matrix

	λ_1^*	λ_2^*	λ_3^*
λ_1	1	1	1
λ_2	1	1	0
λ_3	1	0	0
λ_4	1	0	0

This matrix has row sums λ_i and column-sums λ_j^*; the latter fact follows from the fact that the Young diagram of the transpose of a partition is the transpose of its Young diagram, and the column-sums of the matrix are the row-sums of its transpose.

The existence of the aforementioned matrix shows that $a_{\lambda,\lambda^*} \geq 1$. However, this is the only such matrix possible. Assume that λ and λ^* have been written in decreasing order (as we have been doing). The first column of the matrix must have λ_1^* 1s, which is the number of λ_i; then the entire first column must be filled with 1s. Then the second columns must have λ_2^* 1s, which is the number of λ_i that are greater than 1; then these all must be filled with 1s. Then at each step, we have that column j must be filled from the top downwards, filling exactly the rows i such that $\lambda_i \geq j$. This produces the matrix described above, so $a_{\lambda,\lambda^*} = 1$.

(3) We use a "rolling ball" argument. Suppose μ is a partition with $a_{\lambda,\mu} > 0$; we show that $\mu \leq \lambda^*$. Since $a_{\lambda,\mu} > 0$, there exists some 1-0 matrix with row-sums λ_i and column-sums μ_i; construct this matrix. For example, if $\lambda = (3,2,1,1)$ as before, and $\mu = (3,2,1,1)$, then we can construct the matrix

$$\begin{array}{c|cccc} & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \hline \lambda_1 & 1 & 1 & 1 & 0 \\ \lambda_2 & 1 & 0 & 0 & 1 \\ \lambda_3 & 0 & 1 & 0 & 0 \\ \lambda_4 & 1 & 0 & 0 & 0 \end{array}$$

What does $\mu \leq \lambda^*$ mean in this context? It means that for each k, the sum of the first k columns of the λ, λ^* matrix is greater than sum of the 1st k columns of the λ, μ matrix. Imagine the 1s as balls on shelves (the rows). Tilting the matrix counterclockwise to make the balls roll to the left will only make the sum of the first k columns larger (for each k). But doing so gives us the λ, λ^* matrix. So $\mu \leq \lambda^*$, and we are done. $\square$

Corollary 1.1. *The e_λ form a basis for the symmetric functions.*

Proof. The previous proposition gives us that

$$e_\lambda = m_{\lambda^*} + \sum_{\mu < \lambda^*} a_{\lambda,\mu} m_\mu.$$

Solving for m_{λ^*} gives us that

$$m_{\lambda^*} = e_\lambda + \sum_{\mu < \lambda^*} -a_{\lambda,\mu} m_{\mu^*}.$$

There is a unique minimal element with respect to the partial order on partitions of n, $\lambda = (1,1,\ldots,1)$. Then, given that we can express m_μ in terms of elementary symmetric functions for $\mu \leq \lambda$, we can express m_λ in terms of elementary symmetric functions. So all the m_λ can be expressed in terms of elementary symmetric functions, and we are done. $\square$